Gaudin models and bending flows: a geometrical point of view

2003 J. Phys. A: Math. Gen. 36 11655

(http://iopscience.iop.org/0305-4470/36/46/009)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 11655–11676 PII: S0305-4470(03)64456-2

Gaudin models and bending flows: a geometrical point of view

Gregorio Falqui and Fabio Musso

SISSA, Via Beirut 2/4, I-34014 Trieste, Italy

E-mail: falqui@sissa.it and musso@sissa.it

Received 5 June 2003, in final form 28 August 2003
Published 5 November 2003
Online at stacks.iop.org/JPhysA/36/11655

Abstract
In this paper, we discuss the bi-Hamiltonian formulation of the (rational XXX) Gaudin models of spin–spin interaction, generalized to the case of $sl(r)$-valued 'spins'. We only consider the classical case, using recent results concerning the quantum models as guiding principles. In particular, we focus on the so-called homogeneous XXX models. We find a pencil of Poisson brackets that recursively define a complete set of integrals of the motion, alternative to the set of integrals associated with the 'standard' Lax representation of the Gaudin model. These integrals, in the case of $su(2)$, coincide with the Hamiltonians of the 'bending flows' in the moduli space of polygons in the Euclidean space introduced by Kapovich and Millson. We finally address the problem of separability of these flows and explicitly find separation coordinates and separation relations for the $sl(2)$ case.

PACS numbers: 45.20.Jj, 02.30.Ik, 75.10.Pq
Mathematics Subject Classification: 70H06, 37K10, 70H20

1. Introduction

In [13], Gaudin proved the integrability of N-site $su(2)$ (quantum) spin Hamiltonians of the form

$$\mathcal{H} = \sum_{j<l=1}^{N} \frac{c_j - c_l}{a_j - a_l} \left(\sigma_j^x \sigma_l^x + \sigma_j^y \sigma_l^y + \sigma_j^z \sigma_l^z\right) \tag{1.1}$$

for any choice of the arbitrary parameters a_j and c_j ($a_i \neq a_j, c_i \neq c_j, i \neq j$). Here, $\mathcal{H}$ is an operator acting on the Hilbert space of states of the model,

$$\mathcal{V} = V_1 \otimes V_2 \otimes \cdots \otimes V_N$$

where V_j is a copy of the spin 1/2 representation of $SU(2)$, $\sigma^x, \sigma^y, \sigma^z$ are standard Pauli matrices and, e.g., σ_j^x stands for the operator

$$\sigma_j^x = \mathbf{1} \otimes \cdots \underbrace{\otimes\, \sigma^x \otimes}_{j\text{th place}} \mathbf{1} \cdots \otimes \mathbf{1}.$$

The property of integrability follows from the fact that one can write the Gaudin Hamiltonian $\mathcal{H}$ as

$$\mathcal{H} = \sum_{j=1}^{N} c_j H_j \qquad \text{with} \qquad H_j = \sum_{l \neq j} \frac{1}{a_j - a_l} \left(\sigma_j^x \sigma_l^x + \sigma_j^y \sigma_l^y + \sigma_j^z \sigma_l^z \right) \tag{1.2}$$

and check that the H_j define a set of $N - 1$ commuting observables. Since fixing the values of the N Casimirs

$$C_j = \sigma_j^x \sigma_j^x + \sigma_j^y \sigma_j^y + \sigma_j^z \sigma_j^z$$

the system has N degrees of freedom, the $N - 1$ quantities H_j, together with, e.g., $S_z = \sum_{i=1}^N \sigma_i^z$, provide a complete set of mutually commuting observables.

A relevant member of this class of Hamiltonians, whose classical counterpart will be the main subject of the present paper, is obtained when one chooses c_k to be proportional to a_k for all k, so that, up to a rescaling, the Hamiltonian (1.1) becomes

$$\mathcal{H} = \frac{1}{2} \sum_{j,l=1}^{N} \left(\sigma_j^x \sigma_l^x + \sigma_j^y \sigma_l^y + \sigma_j^z \sigma_l^z \right). \tag{1.3}$$

Following [3] we will refer to the model described by $\mathcal{H}$ as the *XXX rational homogeneous*[1] Gaudin model.

This system is not only integrable, but maximally super-integrable. One can understand this stronger property as follows (see, e.g [16]): since the 'physical' Hamiltonian (1.3) is independent of the parameters, the choice of the a_k in the definition of the commuting integrals H_j is arbitrary (provided $a_i \neq a_j, i \neq j$). So, choosing another set of parameters $b_k \neq a_k$ and considering $\tilde{H}_l = \sum_{l \neq j} \frac{1}{b_j - b_l} \left(\sigma_j^x \sigma_l^x + \sigma_j^y \sigma_l^y + \sigma_j^z \sigma_l^z \right)$ one can define the two sets of complete commuting quantities:

$$\{\mathcal{H}, H_1, \ldots, H_{N-2}, S_z\} \qquad \{\mathcal{H}, \tilde{H}_1, \ldots, \tilde{H}_{N-2}, S_x\}.$$

Since for generic choices of the sets a_k, b_k the observables

$$\{\mathcal{H}, H_1, \ldots, H_{N-2}, \tilde{H}_1, \ldots, \tilde{H}_{N-2}, S_z, S_x\}$$

are algebraically independent, the model is indeed maximally super-integrable.

Recently it was pointed out independently by various authors [5, 19], that with (1.3) it is possible to associate a set of commuting integrals *independent* of the parameters. Such operators are of the form:

$$\mathcal{I}_{k-1} = \frac{1}{2} \sum_{j,l=1}^{k} \left(\sigma_j^x \sigma_l^x + \sigma_j^y \sigma_l^y + \sigma_j^z \sigma_l^z \right) \qquad k = 2, \ldots, N \tag{1.4}$$

and, together with S_z they form a complete set of involutive integrals for $\mathcal{H}$.

The classical counterpart of the $su(2)$ XXX homogeneous Gaudin model is the following Hamiltonian system. We consider as phase space the N-fold Cartesian product of the Lie–Poisson manifold associated with $su(2)$ which we parametrize by means of N Hermitian

[1] The name homogeneous refers to the fact that the interaction between the spins is the same for all pairs of interacting 'sites'. Possibly an equally good name could be 'uniform Gaudin model'.

2×2 matrices $A_1, \ldots, A_n$. The entries of such matrices can be seen as coordinates on our Poisson manifold. On this phase space, the Gaudin Hamiltonian is given by $H = \frac{1}{2}\sum_{j\neq l=1}^{N} \mathrm{Tr}(A_j \cdot A_l)$. The (Hamilton) equation of motion associated with H via the standard Lie–Poisson structure (see section 2.1) has the simple Lax form:

$$\frac{\mathrm{d}}{\mathrm{dt}} A_l = \mathrm{i}\left[A_l, \sum_{j=1}^{N} A_j\right] \qquad l = 1, \ldots, N \qquad \text{where} \quad \mathrm{i} = \sqrt{-1}. \tag{1.5}$$

The property of integrability (and super-integrability) of this classical system can be proved *mutatis mutandis* exactly as that of its quantum counterpart, considering the quantities $H_j = \sum_{l\neq j} \frac{\mathrm{Tr}(A_l \cdot A_j)}{a_j - a_l}$, $\tilde{H}_j = \sum_{l\neq j} \frac{\mathrm{Tr}(A_l \cdot A_j)}{b_j - b_l}$ and the global invariance under the group $SU(2)$.

The classical version of the integrals (1.4), namely

$$I_{k-1} = \frac{1}{2}\mathrm{Tr}\left(\left(\sum_{i=1}^{k} A_i\right)^2\right) \qquad k = 2, \ldots, N \tag{1.6}$$

was recently considered, in a completely different context, by Kapovich and Millson [18]. These authors (see also [12]) studied the moduli space of $(N+3)$-sided polygons in $\mathbb{R}^3$, and (implicitly) showed that it coincides with a suitable Marsden–Weinstein quotient (with respect to the diagonal action of $SU(2)$) of the phase space of the $N+3$ site $su(2)$-Gaudin models. They remarked that such a space possesses a natural Hamiltonian structure, and integrated, via action-angle variable methods, the flows associated with the integrals I_{k-1}, $k = 2, \ldots, N$. It is worthwhile to remind the intriguing representation of such flows: if one draws, from a chosen vertex, the N possible diagonals of an $(N+3)$-sided polygon, the flow associated with the Hamiltonian I_k geometrically represents the bending of one side of the polygon along the kth diagonal (the other side being kept fixed), whence the name of 'bending flows'.

The Gaudin system (1.1) admits various generalizations. Gaudin himself pointed out that the integrals (1.2) can be generalized to any semisimple Lie algebra $\mathfrak{g}$. Clearly, if the rank of $\mathfrak{g}$ is greater than 1, the number of such integrals is not enough to ensure complete integrability. The missing integrals have been shown by Jurčo [17] and Sklyanin [29] to be provided by the spectral invariants of a suitable Lax matrix, whose classical counterpart is

$$L_{\mathrm{rat}} = \sum_{i=1}^{N} \frac{A_i}{\lambda - a_i} \tag{1.7}$$

where $a_i \neq a_j$, $i \neq j$ and the A_i are generic elements of $\mathfrak{g}$. In terms of the Lax matrix (1.7) the generalization of the Hamiltonian (1.3) reads

$$H_G = \sum_{i=1}^{N} \mathrm{res}_{|\lambda=a_i} \frac{1}{2}\mathrm{Tr}\big(\lambda L_{\mathrm{rat}}^2\big) = \frac{1}{2}\sum_{j\neq i} \mathrm{Tr}(A_i A_j). \tag{1.8}$$

Another straightforward generalization of this model is obtained by adding a constant matrix κ to the Lax matrix, yielding

$$L_{\mathrm{rat}}^{(\kappa)} = \kappa + \sum_{i=1}^{N} \frac{A_i}{\lambda - a_i}. \tag{1.9}$$

In the $su(2)$ case, this is equivalent to adding to the Hamiltonian (1.3) a term describing the interaction of the spins with a magnetic field with a constant direction in each site but with different intensity, that is, to consider the Hamiltonian

$$H_G' = \frac{1}{2}\sum_{j\neq i} \mathrm{Tr}(A_i A_j) + \sum_i a_i \,\mathrm{Tr}(\kappa \cdot A_i). \tag{1.10}$$

We will call this function the Hamiltonian of the 'inhomogeneous' (XXX) Gaudin magnet. The complete integrability and separability of these systems, (for the $\mathfrak{g} = sl(n)$ case) were studied and proved in [30, 14, 28].

The aim of this paper is to frame the analysis of the Gaudin models, as well as of the Hamiltonians (1.6) of the bending flows of Kapovich and Millson, in the scheme of bi-Hamiltonian geometry as advocated by Gel'fand and Zakharevich [15], and to show how one can use this scheme to explicitly integrate the model for $\mathfrak{g} = sl(2)$. We will consider only the classical models, and consider the complexified case (that is, we will study the Gaudin system associated with a complex semisimple Lie algebra $\mathfrak{g}$).

Our first task will be to briefly show how, using nowadays standard results of the theory of r-matrices on loop algebras (see, e.g., [27]), one can provide the phase space of the (inhomogeneous) Gaudin magnet with a bi-Hamiltonian structure, selecting it out of a multi-parameter family of Poisson structures. This structure gives rise, according to the Gel'fand–Zakharevich (GZ) scheme, to the integrals associated with the Lax matrices of Jurčo and Sklyanin (1.7), (1.9).

Then we will construct, in the homogeneous case, *another* bi-Hamiltonian structure, non-compatible (in a sense to be precised later) with the above-mentioned family, whose GZ analysis gives rise, in the $sl(2)$ case, to the parameter-independent integrals (1.6). Since such additional bi-Hamiltonian structure is still constructed within a Lie-theoretical setting, we will be able to straightforwardly apply this scheme to $\mathfrak{g} = sl(r)$, with arbitrary r. In this way, we will be able to find a sufficient number of commuting integrals to be added to the 'generalized bending Hamiltonians' I_k, yielding a complete family of integrals alternative to the 'standard' family obtained by Sklyanin and Jurčo.

The GZ analysis of such a model will finally lead us to introduce a kind of Lax matrices for such flows and to show that the Hamilton–Jacobi equations associated with the $sl(2)$ bending Hamiltonians are separable by computing explicitly the separation variables and the separation relations.

2. GZ analysis of Gaudin models

The GZ scheme [15] for integrating a bi-Hamiltonian system can be seen as a particularly efficient scheme to implement the Lenard–Magri recursion for manifolds endowed with a pair of compatible Poisson brackets none of which is symplectic (i.e., non-degenerate).

One considers a manifold M endowed with a pair of compatible Poisson tensors $P_1 - \lambda P_0$, or, in other words, a pencil of Poisson brackets

$$\{f, g\}_\lambda = \{f, g\}_{P_1} - \lambda\{f, g\}_{P_0} = \langle \mathrm{d}f, (P_1 - \lambda P_0)\,\mathrm{d}g\rangle \tag{2.1}$$

(where $\langle\cdot,\cdot\rangle$ is the canonical pairing between T^*M and TM), and assumes that the kernel of the generic element of the Poisson pencil is k dimensional. Let $C_1, \ldots, C_k$ be independent Casimir functions of P_0. The GZ method, roughly speaking, suggests to use these Casimirs as 'starting' elements for Lenard chains yielding (under some technical additional conditions), via the method of bi-Hamiltonian iteration, families of functions $\left\{H_m^{(a)}\right\}_{a=1,\ldots,k}^{m=0,\ldots}$, such that for any function F on M, and $a = 1, \ldots, k$,

$$\left\{F, H_m^{(a)}\right\}_{P_0} = \left\{F, H_{m-1}^{(a)}\right\}_{P_1} \qquad \text{with} \qquad H_0^{(a)} = C_a. \tag{2.2}$$

As a consequence of the bi-Hamiltonian iterative scheme and of the fact that all Lenard chains start with a Casimir function of P_0 (they are 'anchored', in the language of [15]), all these functions are mutually in involution with respect to both Poisson brackets. Obviously, the maximal number of independent functions one may hope to get in this way is $N_{\max} = \frac{1}{2}(\dim M + k)$. If this is indeed the case, the geometric scheme hereafter outlined defines families of completely integrable systems in the Liouville sense. Indeed, let us suppose that the GZ method provides us with k families of mutually commuting independent functions

$$\left\{H_m^{(a)}\right\}_{a=1,\ldots,k}^{m=0,\ldots,n_a} \qquad \text{with} \qquad \sum_{a=1}^{k} n_a = \frac{1}{2}(\dim M + k).$$

Let $\mathcal{H}$ be a generic element in the ring generated by such commuting functions, and let $X_{\mathcal{H}} = P_0\,\mathrm{d}\mathcal{H}$ be the corresponding Hamiltonian vector field. Let us consider a generic symplectic leaf $S \subset M$ of P_0; it is a $d_s = \dim M - k$ dimensional manifold, with the natural symplectic form induced by the Poisson structure P_0. $X_{\mathcal{H}}$ clearly restricts to S, and, as a consequence of the bi-Hamiltonian iteration on M, comes equipped with $\frac{1}{2}(\dim M + k) - k = \frac{1}{2}d_S$ integrals in involution, given by the restriction to S of the functions $\left\{H_m^{(a)}\right\}_{a=1,\ldots,k}^{m=1,\ldots,n_a}$. As a consequence of the genericity assumption on the symplectic leaves, these functions will be independent of S as well and give the complete family of involutive integrals required by the Liouville theorem.

The aim of this section is to frame the (general, that is, inhomogeneous) Gaudin model within the bi-Hamiltonian scheme, and to reinterpret its complete integrability within the theoretical framework of the GZ analysis briefly sketched above. The manifolds we will consider will be Cartesian products of N copies of a Lie algebra $\mathfrak{g}$, and the Poisson structures will always be *linear* structures on $\mathfrak{g}^N$. We will frame our study in the general scheme concerning the multi-Hamiltonian structure of polynomial pencils of matrices that can be found in [27]. In particular, in this section we will constantly rely on the commutativity property of the spectral invariants of such polynomial pencils of matrices, whose proof can be found in the above-mentioned paper. Some details of such a theory closely related to the bi-Hamiltonian approach pursued in the present paper can be found in [26].

We will hereafter mostly limit ourselves to state results and sketch proofs. Our main task will be to choose a specific bi-Hamiltonian structure[2] for the Gaudin models, and to explicitly study the associated GZ polynomials, generalizing to the N particle and arbitrary simple $\mathfrak{g}$ case the results, exposed in [11], concerning the three-particle $sl(2)$-case.

2.1. *Notation and conventions*

Let us briefly recall the notion of Lie–Poisson brackets associated with a Lie algebra and fix some notation and conventions we will use throughout the paper.

If $\mathfrak{g}$ is a Lie algebra, its dual $\mathfrak{g}^*$ comes equipped with the standard Lie–Poisson structure:

$$\{F, G\}(A) = \langle A, [\mathrm{d}F, \mathrm{d}G]\rangle = \langle \mathrm{d}F, P\,\mathrm{d}G\rangle \qquad F, G \in C^\infty(\mathfrak{g}^*). \tag{2.3}$$

If $\mathfrak{g}$ is semisimple we can identify $\mathfrak{g}^*$ with $\mathfrak{g}$. Indeed, we can associate a matrix X_A with any element $A \in \mathfrak{g}^*$ considering, e.g., the fundamental representation of the algebra $\mathfrak{g}$, and taking the trace form as a bilinear non-degenerate pairing

$$\langle A, X_B\rangle = \mathrm{Tr}(X_A \cdot X_B).$$

From now on we will implicitly use this identification, and write $A, B\ldots$ instead of $X_A, X_B, \ldots$ for simplicity of notation. Using the cyclicity of the trace, the Hamiltonian vector field associated by (2.3) with a smooth function F is represented by

$$P\,\mathrm{d}F = \dot{A} = \left[A, \frac{\partial F}{\partial A}\right]$$

[2] A different scheme to provide Gaudin models with a bi-Hamiltonian structure has recently appeared in the literature [25].

where the symbol $\frac{\partial F}{\partial A}$ denotes the matrix satisfying, for any Ξ in $\mathfrak{g}$,

$$F(A + t\Xi) = F(A) + t \cdot \mathrm{Tr}\left(\frac{\partial F}{\partial A} \cdot \Xi\right) + o(t).$$

If we take the direct product of N copies of $\mathfrak{g}$, the standard Lie–Poisson structure becomes

$$\{F, G\}(A_1, \ldots, A_N) = \sum_{i=1}^{N} \left\langle A_i, \left[\frac{\partial F}{\partial A_i}, \frac{\partial G}{\partial A_i}\right]\right\rangle \tag{2.4}$$

and the Hamiltonian vector field associated with a function F is

$$\dot{A}_i = \left[A_i, \frac{\partial F}{\partial A_i}\right] \qquad i = 1, \ldots, N.$$

We can write the above equation in the form

$$\frac{\partial A_i}{\partial t} = (X_F)_i = (P\,\mathrm{d}F)_i = \sum_{j,k} p_{ijk}\left[A_k, \frac{\partial F}{\partial A_j}\right] \qquad \text{with} \qquad p_{ijk} = \delta_{ij}\delta_{jk}. \tag{2.5}$$

We will also often write P (and other Poisson tensors) representing its action on the differential of a function by means of the matrix symbolic form:

$$\begin{pmatrix} \dot{A}_1 \\ \dot{A}_2 \\ \vdots \\ \dot{A}_N \end{pmatrix} = \begin{pmatrix} [A_1, .] & 0 & \ldots & 0 \\ 0 & [A_2, .] & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & [A_N, .] \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial F}{\partial A_1} \\ \dfrac{\partial F}{\partial A_2} \\ \vdots \\ \dfrac{\partial F}{\partial A_N} \end{pmatrix}. \tag{2.6}$$

For this reason, we will term the standard Lie–Poisson tensor P on $\mathfrak{g}^N$ the *diagonal* Poisson tensor.

2.2. A bi-Hamiltonian structure of the Gaudin model

A bi-Hamiltonian structure for rational Gaudin models can be obtained using the following argument. Let us consider the map $\{A_i\} \longrightarrow \{B_i\}$ that sends the rational Lax matrix

$$L_{\mathrm{rat}}^{(\kappa)} = \kappa + \sum_{i=1}^{N} \frac{A_i}{\lambda - a_i}$$

in the polynomial Lax matrix

$$L_{\mathrm{poly}}^{(\kappa)} = \lambda^N \kappa + \sum_{i=0}^{N-1} B_i \lambda^i = \left(\prod_{i=1}^{N} (\lambda - a_i)\right) \cdot L_{\mathrm{rat}}^{(\kappa)} \tag{2.7}$$

given explicitly by

$$\begin{aligned} B_l = (-1)^{N-l-1} \sum_{i=1}^{N} s_{N-l-1}(a_1, \ldots, \hat{a}_i, \ldots, a_N) \cdot A_i + (-1)^{N-l} s_{N-l}(a_1, \ldots, a_N) \cdot \kappa \\ l = 0, \ldots, N-1 \end{aligned} \tag{2.8}$$

where $s_k(a_1, \ldots, a_N)$ denotes the kth elementary symmetric polynomial in the variables $a_1, \ldots, a_N$.

On the space of polynomial pencils of matrices a family of mutually compatible Poisson brackets are defined [27, 20]. They will be termed, for the sake of brevity, RSTS tensors. In a nutshell, this family can be described by stating that there is a map from degree N polynomials in the variable λ to the set of Poisson structures on the manifold of polynomial Lax matrices of the form (2.7) which sends the monomials $\lambda^0, \ldots, \lambda^N$ into $N + 1$ fundamental Poisson brackets, $\Pi_l, l = 0, \ldots, N$. In our case, the fundamental tensors Π_l can be represented by matrices having the following block-diagonal structure:

$$\Pi_l = \begin{pmatrix} C_l & 0 \\ 0 & D_l \end{pmatrix} \tag{2.9}$$

with

$$\begin{cases} (C_l)_{ij} = -[B_{i+j-l-1}, \cdot] & i, j = 1, \ldots, l \\ (D_l)_{ij} = [B_{i+j+l-1}, \cdot] & i, j = 1, \ldots, N - l \end{cases} \tag{2.10}$$

$$B_i = 0 \qquad \text{if} \quad i < 0 \quad \text{or} \quad i > N \qquad \text{and} \qquad B_N = \kappa.$$

Lemma 1. *In the 'coordinates' $B_0, \ldots, B_{N-1}, \kappa$, the diagonal Poisson tensor P (2.6) is given by the sum*

$$P = \sum_{l=0}^{N} (-1)^{N-l-1} s_{N-l}(a_1, \ldots, a_N)\Pi_l \tag{2.11}$$

where the s_i are the elementary symmetric polynomials in the a_i, that is, it is the tensor associated with the polynomial

$$p_N = \prod_{i=1}^{N} (\lambda - a_i).$$

This lemma can be proved by means of a direct computation. For the reader's convenience, we collect its main steps in appendix A.

Since the Poisson tensors (2.9) form a $(N + 1)$-parameter family of compatible Poisson tensors, we can choose as a second Poisson tensor a suitable linear combination of them to have a bi-Hamiltonian structure on $\mathfrak{g}^N$. Let

$$Q = \sum_{l=0}^{N-1} (-1)^{N-l} s_{N-l-1}(a_1, \ldots, a_N)\Pi_l \tag{2.12}$$

be the tensor associated with the polynomial

$$p_{N-1} = \left(\frac{p_N}{\lambda}\right)_+ = \lambda^{N-1} - s_1\lambda^{N-2} + \cdots + (-1)^N s_{N-1}.$$

All the integrals of motion that one can obtain from the spectral invariants of the Lax matrix (1.9) can be obtained by the GZ method applied to the pencil $Q - \lambda P$; in fact it holds (see, also, [27]).

Lemma 2. *All the vector fields associated with the spectral invariants of (1.9) are bi-Hamiltonian with respect to the pair $Q - \lambda P$.*

Proof. We find convenient to work in the variables B_i. Let us define:

$$K_\alpha^{(i)} = \mathrm{Tr}\left(\mathrm{Res}_{\lambda=0}\left(\frac{\left(\sum_{j=1}^{N} B_j\lambda^j\right)^\alpha}{\lambda^i}\right)\right) \qquad i = 1, \ldots, \alpha N \qquad \alpha = 2, \ldots, \mathrm{rk}(\mathfrak{g}). \tag{2.13}$$

For any fixed α, the αN functions (2.13) fulfil the relations [27]:

$$\Pi_i \,\mathrm{d}K^{(j)} = \Pi_{i+k} \,\mathrm{d}K^{(j+k)} = X^{(j-i)}. \tag{2.14}$$

From (2.14) it follows that $X^{(i)} = 0$ if $i \leqslant 0$ or $i > N(\alpha - 1)$; in fact, if $i \leqslant 0$ then $K^{(i)} = 0$ and $X^{(i)} = \Pi_0 \,\mathrm{d}K^{(i)} = 0$, while if $i > N(\alpha - 1)$, then $K^{(N+i)} = \text{const}$ and $X^{(i)} = \Pi_N \,\mathrm{d}K^{(N+i)} = 0$. Now let us set

$$b_l = (-1)^{N-l+1} s_{N-l}(a_1, \ldots, a_N) \tag{2.15}$$

we have

$$P \,\mathrm{d}K^{(j)} = \sum_{l=0}^{N} b_l \Pi_l \,\mathrm{d}K^{(j)} = \sum_{l=0}^{N} b_l X^{(j-l)}$$

$$Q \,\mathrm{d}K^{(j)} = \sum_{l=1}^{N} b_l \Pi_{l-1} \,\mathrm{d}K^{(j)} = \sum_{l=1}^{N} b_l X^{(j-l-1)}.$$

Then

$$P \,\mathrm{d}K_\alpha^{(j)} - Q \,\mathrm{d}K_\alpha^{(j+1)} = b_0 X_\alpha^{(j)}.$$

If one of the a_i is equal to zero, then $b_0 = \prod_{i=1}^{N} a_i = 0$ and the proof is concluded. Otherwise we need to find a function $F_\alpha^{(j)}$ such that

$$Q \,\mathrm{d}F_\alpha^{(j)} = b_0 X_\alpha^{(j)}.$$

We proceed by induction. If $j = 1$, we have $b_0 X_\alpha^{(1)} = Q \frac{b_0}{b_1} \,\mathrm{d}K_\alpha^{(1)}$, so that $F_\alpha^{(1)} = b_0 \big/ b_1 K_\alpha^{(1)}$. Now let $F_\alpha^{(i)}$ be such that: $b_0 X_\alpha^{(i)} = Q \,\mathrm{d}F_\alpha^{(i)}$, $i = 1, \ldots, j-1$. Then

$$Q \frac{b_0}{b_1} \,\mathrm{d}K_\alpha^{(j)} = b_0 X_\alpha^{(j)} + \frac{b_0 b_2}{b_1} X_\alpha^{(j-1)} + \cdots + \frac{b_0 b_N}{b_1} X_\alpha^{(j-N+1)}$$
$$\Longrightarrow \quad b_0 X_\alpha^{(j)} = Q \left(\frac{b_0}{b_1} \,\mathrm{d}K_\alpha^{(j)} - \frac{b_2}{b_1} \,\mathrm{d}F_\alpha^{(j-1)} - \cdots - \frac{b_N}{b_1} \,\mathrm{d}F_\alpha^{(j-N+1)} \right).$$

So we have

$$Q \,\mathrm{d}F_\alpha^{(j)} = b_0 X_\alpha^{(j)} \qquad \text{with} \qquad F_\alpha^{(j)} = \frac{b_0}{b_1} K_\alpha^{(j)} - \frac{1}{b_1} \sum_{i=1}^{N-1} b_{i+1} F_\alpha^{(j-i)}. \tag{2.16}$$

□

Some observations on the GZ sequences are in order. The starting points of the GZ sequences are given by the Casimirs of P. We have to distinguish two cases:

(a) If $b_0 \neq 0$, i.e. all the a_i are different from zero, then the Casimirs of P are given in terms of the spectral invariants (2.13) by the following expressions:

$$C_{i,\alpha} = \sum_{j=1}^{\alpha N} a_i^j K_\alpha^{(j)} \qquad i = 1, \ldots, N \qquad \alpha = 2, \ldots, \mathrm{rk}(\mathfrak{g}). \tag{2.17}$$

For any α, starting the GZ sequences from suitable linear combinations of the Casimirs $C_{i,\alpha}$ we can construct N GZ sequences of length $\alpha - 1$ (i.e. defining $\alpha - 1$ independent vector fields) each starting with a Casimir of P and ending with a Casimir of Q.

(b) If $b_0 = 0$ then only one among the a_i, say a_N, is zero. In this case, equation (2.17) defines $(\mathrm{rk}(\mathfrak{g}) - 1)(N - 1)$ independent Casimirs, instead of $(\mathrm{rk}(\mathfrak{g}) - 1)N$:

$$C_{i,\alpha} = \sum_{j=1}^{\alpha N} a_i^j K_\alpha^{(j)} \qquad i = 1, \ldots, N-1 \qquad \alpha = 2, \ldots, \mathrm{rk}(\mathfrak{g}). \tag{2.18}$$

The functions (2.18) turns out to be simultaneous Casimirs for both P and Q. The remaining $\mathrm{rk}(\mathfrak{g}) - 1$ Casimirs of P (the rank of P is obviously the same in both cases) are given by

$$C_{N_\alpha} = K_\alpha^{(1)} \qquad \alpha = 2, \ldots, \mathrm{rk}(\mathfrak{g}). \tag{2.19}$$

With each Casimir (2.19) is associated a GZ sequence of length $(\alpha - 1)N$.

3. The homogeneous case

The constant term κ in the Lax matrix (1.9) physically describes the coupling of the ith spin with an 'external magnetic' field $\beta_i = a_i \kappa$. The matrix κ in the definition of the rational Lax matrix (1.9) is somewhat a free parameter in the theory. Changing κ amounts to 'changing the direction' of this magnetic field. The choice usually made in the literature is the generic one (say, κ is a diagonal matrix with different entries); this ensures the functional independence of the coefficients of the spectral invariants of $L_{\mathrm{rat}}^{(\kappa)}$, whence the fact that they are in a sufficient number to yield complete integrability of the model.

If κ is not generic, but the dimension of its stabilizer $\mathfrak{g}_\kappa := \{\tau \in \mathfrak{g} \text{ s.t. } [\tau, \kappa] = 0\}$ is greater than the rank of $\mathfrak{g}$ the following happens. Not all the spectral invariants of the Lax matrix are functionally independent, but one can recover the 'missing' integrals noting that the functions:

$$F_\tau = \mathrm{Tr}\left(\sum_{i=1}^{N} A_i \tau\right) \qquad \tau \in \mathfrak{g}_\kappa$$

commute with all the spectral invariants of $L_{\mathrm{rat}}^{(\kappa)}$.

However, something more substantial occurs for $\kappa = 0$, that is, in the homogeneous case. As we have recalled in the introduction, in such a case H_G (1.8) defines, for $\mathfrak{g} = sl(2)$, a super-integrable Hamiltonian system and, in particular, it is possible to find another complete set of commuting first integrals which are not explicitly dependent on the parameters a_i.

From now on we will focus on this additional family of integrals, that, in the classical N-site $sl(2)$ model can be given by the very simple formula:

$$I_{l-1} = \sum_{j,k=1}^{l} \mathrm{Tr}(A_j \cdot A_k) \qquad l = 2, \ldots, N. \tag{3.1}$$

We will introduce further a Poisson structure R on the manifold $\mathfrak{g}^N$. As we shall show it will be possible to combine it with the diagonal Poisson structure P of equation (2.6) to get a further *Poisson pencil*, not belonging to the RSTS family described in section 2. The GZ method applied to the Poisson pencil $R - \lambda P$ will give rise to these new sets of integrals. Since everything will be done in a Lie-algebraic setting, these results hold for a generic semisimple Lie algebra, and in particular, for $sl(r)$ with arbitrary r.

3.1. The additional bi-Hamiltonian pencil

Let us consider the bivector R, defined, by means of the constructions outlined in section 2 by the following matrix:

$$R = \begin{pmatrix} 0 & [A_1, \cdot] & \cdots & [A_1, \cdot] \\ [A_1, \cdot] & [A_2 - A_1, \cdot] & \cdots & [A_2, \cdot] \\ \vdots & \vdots & \ddots & \vdots \\ [A_1, \cdot] & [A_2, \cdot] & \cdots & [(N-1)A_N - \sum_{i=1}^{N-1} A_i, \cdot] \end{pmatrix}. \tag{3.2}$$

Proposition 3.1. *The bivector R defined by (3.2) is a Poisson bivector, and it is compatible with the diagonal Poisson tensor P.*

Proof. Linearity and antisymmetry are obvious, so we must prove only the Jacobi identity. Also, we can limit ourselves to prove the assertions for the case of linear functions on $\mathfrak{g}$. If F, G, H are such functions, identifying their differentials with the three N-tuples of matrices $\{\alpha_i\}, \{\beta_i\}, \{\gamma_i\}$, (e.g., $\frac{\partial F}{\partial A_i} = \alpha_i, \ldots$), the Poisson bracket is defined by

$$\{F, G\}_R = \langle \mathrm{d}F, R\,\mathrm{d}G\rangle = \sum_{i,j,k} r_{ijk}\,\mathrm{Tr}(\alpha_i[A_k, \beta_j]) = \sum_{i,j,k} r_{ijk}\,\mathrm{Tr}(A_k[\beta_j, \alpha_i])$$
$$r_{ijk} = (k-1)\delta_{ij}\delta_{jk} - \theta_{(i-k)}\delta_{ij} + \theta_{(j-i)}\delta_{ik} + \theta_{(i-j)}\delta_{jk}$$

where δ is the usual Kronecker symbol and θ is the discrete Heaviside function, normalized as

$$\theta_{(i)} = \begin{cases} 1 & \text{if} \quad i > 0 \\ 0 & \text{if} \quad i \leqslant 0. \end{cases}$$

The Jacobi identity reads

$$\{H, \{F, G\}_R\}_R + \{F, \{G, H\}_R\}_R + \{G, \{H, F\}_R\}_R = \sum_{i,j,k,l,m} r_{ijk}r_{lmj}(\mathrm{Tr}(A_k[[\beta_m, \alpha_l], \gamma_i]) + \mathrm{Tr}(A_k[[\alpha_m, \gamma_l], \beta_i]) + \mathrm{Tr}(A_k[[\gamma_m, \beta_l], \alpha_i]))$$

which, renaming the indices, becomes

$$\sum_{i,j,k,l,m} r_{ijk}r_{lmj}\,\mathrm{Tr}(A_k[[\beta_m, \alpha_l], \gamma_i]) + r_{mjk}r_{ilj}\,\mathrm{Tr}(A_k[[\alpha_l, \gamma_i], \beta_m]) + r_{ljk}r_{mij}\,\mathrm{Tr}(A_k[[\gamma_i, \beta_m], \alpha_l]).$$

A sufficient condition for the last expression to be zero is that for any k it holds that

$$\sum_j r_{ijk}r_{lmj} = \sum_j r_{mjk}r_{ilj}. \tag{3.3}$$

In fact, a consequence of (3.3) to hold implies that $t_{iklm} = \sum_j r_{ijk}r_{lmj}$ is invariant for cyclic permutations of the indices i, l, m. So, if (3.3) holds we can write:

$$\{H, \{F, G\}_R\}_R + \{F, \{G, H\}_R\}_R + \{G, \{H, F\}_R\}_R = \sum_{i,k,l,m} t_{iklm}\,\mathrm{Tr}(A_k([[\beta_m, \alpha_l], \gamma_i] + [[\alpha_l, \gamma_i], \beta_m] + [[\gamma_i, \beta_m], \alpha_l]))$$

which vanishes thanks to the Jacobi identity in $\mathfrak{g}$.

Let us show that (3.3) holds in our case. By means of algebraic manipulations, namely cycling through i, l, m and renaming the indices using the Kronecker's δ, we obtain

$$\sum_j r_{ijk}r_{lmj} - \sum_j r_{mjk}r_{ilj} = \delta_{ik}\delta_{lm}\left[(l-i)\theta_{(l-i)}\sum_j \theta_{(j-i)}\theta_{(l-j)}\right] + \delta_{il}\delta_{mk}\left[(i-m)\theta_{(i-m)} + \sum_j \theta_{(j-m)}\theta_{(i-j)}\right] + \delta_{lm}[\theta_{(i-k)}(\theta_{(l-i)} - \theta_{(l-k)}) + \theta_{(l-k)}\theta_{(i-l)}] + \delta_{il}[\theta_{(m-k)}(\theta_{(i-k)} - \theta_{(i-m)}) - \theta_{(i-k)}\theta_{(m-i)}] + \delta_{ik}[\theta_{(l-i)}(\theta_{(m-l)} - \theta_{(m-i)}) + \theta_{(m-i)}\theta_{(l-m)}] + \delta_{mk}[\theta_{(l-m)}(\theta_{(i-m)} - \theta_{(i-l)}) - \theta_{(i-m)}\theta_{(l-i)}].$$

Using the identities

$$\sum_j \theta_{(j-i)}\theta_{(l-j)} = (l-i-1)\theta_{(l-i)}$$

$$\theta_{(i-k)}(\theta_{(l-i)} - \theta_{(l-k)}) + \theta_{(l-k)}\theta_{(i-l)} = -\theta_{(l-k)}\delta_{il}$$

we see that every term cancels out.

We now prove that R is compatible with the diagonal tensor P. The characteristic condition for the compatibility of two Poisson tensors is

$$\{H, \{F, G\}_P\}_R + \{H, \{F, G\}_R\}_P + \text{cyclic permutations of } F, G, H = 0.$$

Recalling that $\{F, G\}_P = \sum_{i,j,k} \delta_{ij}\delta_{jk} \operatorname{Tr}\big(A_k\big[\frac{\partial G}{\partial A_j}, \frac{\partial F}{\partial A_i}\big]\big)$, one shows that a sufficient condition for the compatibility of R and P is that the quantity

$$s_{iklm} = \sum_j (r_{ijk}\delta_{lm}\delta_{mj} + \delta_{ij}\delta_{jk}r_{lmj})$$

be invariant under cyclic permutations of the indices i, l, m for all k. Actually,

$$s_{iklm} = (k+i)\delta_{ik}\delta_{lm}\delta_{il} - \theta_{(i-k)}\delta_{lm}\delta_{il} + \theta_{(i-m)}\delta_{lm}\delta_{kl} + \theta_{(m-l)}\delta_{ik}\delta_{il} + \theta_{(l-m)}\delta_{ik}\delta_{im}$$

manifestly satisfies this property. □

Remark. By the previous proposition, we can endow, for every N, the space $(\mathfrak{g}^*)^N$ with a bi-Hamiltonian structure $P_\lambda = R - \lambda P$. A natural question arises, that is what is the connection with the RSTS family of Poisson structures discussed in section 2. We do not have yet a complete answer to this point; however, as we will show in appendix B, the new tensor R is *not* compatible with the generic element of the RSTS family (2.9).

3.2. The Lenard chains

We construct the Lenard chains for the Poisson pencil $R - \lambda P$, using the GZ recipe discussed in section 2. To shorten the notation we define

$$B_l = \sum_{i=1}^{l-1} A_i \qquad F^{(\alpha)}_{\beta,l} = \operatorname{res}_{\lambda=0} \frac{1}{\lambda^{\alpha+1}} (\lambda A_l + B_l)^\beta \tag{3.4}$$

$$H^{(\alpha)}_{\beta,l} = \operatorname{Tr}\big(F^{(\alpha)}_{\beta,l}\big). \tag{3.5}$$

We will first find a kind of 'modified' recursion relation.

Lemma 3. *It holds that*

$$P\, \mathrm{d}H^{(\alpha-1)}_{\beta,l} = (R - (l-2)P)\, \mathrm{d}H^{(\alpha)}_{\beta,l} \tag{3.6}$$

for $\alpha = 1, \dots, \beta$.

Proof. We proceed by induction on β. If $\beta = 2$ we have

$$H^{(0)}_{2,l} = \operatorname{Tr}\big(B_l^2\big) \quad \Rightarrow \quad \frac{\partial H^{(0)}_{2,l}}{\partial A_j} = 2\theta_{(l-j)} B_l$$

$$H^{(1)}_{2,l} = 2\operatorname{Tr}(A_l B_l) \quad \Rightarrow \quad \frac{\partial H^{(1)}_{2,l}}{\partial A_j} = 2\theta_{(l-j)} A_l + B_l\delta_{jl}$$

$$H^{(2)}_{2,l} = \operatorname{Tr}\big(A_l^2\big) \quad \Rightarrow \quad \frac{\partial H^{(2)}_{2,l}}{\partial A_j} = 2A_l\delta_{jl}.$$

By direct computation we obtain

$$\left((R-(l-2)P)\,\mathrm{d}H^{(2)}_{2,l}\right)_i = 2(\theta_{(l-i)}[A_i, A_l] + \delta_{il}[A_i, B_i]) = \left(P\,\mathrm{d}H^{(1)}_{2,l}\right)_i$$
$$\left((R-(l-2)P)\,\mathrm{d}H^{(1)}_{2,l}\right)_i = 2\theta_{(l-i)}[A_i, B_l] = \left(P\,\mathrm{d}H^{(0)}_{2,l}\right)_i.$$

We use the inductive hypothesis in the case $\alpha \leqslant \beta - 1$. Plugging in the following identities:

$$\left[B_l, F^{(\alpha)}_{\beta-1,l}\right] + \left[A_l, F^{(\alpha-1)}_{\beta-1,l}\right] = 0 \tag{3.7}$$

$$\frac{\partial H^{(\alpha)}_{\beta,l}}{\partial A_j} = \theta_{(l-j)}\frac{\partial H^{(\alpha)}_{\beta,l}}{\partial B_l} + \delta_{jl}\frac{\partial H^{(\alpha)}_{\beta,l}}{\partial A_l} \tag{3.8}$$

$$\frac{\partial H^{(\alpha)}_{\beta,l}}{\partial A_j} = \theta_{(l-j)}F^{(\alpha)}_{\beta-1,l} + B_l\frac{\partial H^{(\alpha)}_{\beta-1,l}}{\partial A_j} + \delta_{jl}F^{(\alpha-1)}_{\beta-1,l} + A_l\frac{\partial H^{(\alpha-1)}_{\beta-1,l}}{\partial A_j} \tag{3.9}$$

$$\frac{\partial H^{(\alpha)}_{\beta,l}}{\partial A_l} = \frac{\partial H^{(\alpha-1)}_{\beta,l}}{\partial B_l} \tag{3.10}$$

and using the inductive hypothesis, one obtains by straightforward computation

$$P\,\mathrm{d}H^{(\alpha-1)}_{\beta,l} - (R-(l-2)P)\,\mathrm{d}H^{(\alpha)}_{\beta,l} = 0.$$

The case $\alpha = \beta$ can be easily verified by direct computation.

Note that identities (3.7) and (3.8) follow from (3.4) and (3.5). The identity (3.9) follows from the recursive formula for $F^{(\alpha)}_{\beta,l}$:

$$F^{(\alpha)}_{\beta,l} = B_l F^{(\alpha)}_{\beta-1,l} + A_l F^{(\alpha-1)}_{\beta-1,l}$$
$$F^{(0)}_{0,l} = 1$$
$$F^{(\alpha)}_{\beta,l} = 0 \qquad \text{if} \quad \alpha > \beta \quad \text{or} \quad \alpha < 0$$

while (3.7) can be proved again by induction. For $\alpha = 2$ it holds that

$$\frac{\partial H^{(2)}_{2,l}}{\partial A_l} = 2A_l = \frac{\partial H^{(1)}_{2,l}}{\partial B_l} \qquad \frac{\partial H^{(1)}_{2,l}}{\partial A_l} = 2B_l = \frac{\partial H^{(0)}_{2,l}}{\partial B_l}.$$

Then, if $\alpha \leqslant \beta - 1$ we can use the inductive hypothesis and get

$$\frac{\partial H^{(\alpha)}_{\beta,l}}{\partial A_l} - \frac{\partial H^{(\alpha-1)}_{\beta,l}}{\partial B_l} = B_l\left(\frac{\partial H^{(\alpha)}_{\beta-1,l}}{\partial A_l} - \frac{\partial H^{(\alpha-1)}_{\beta-1,l}}{\partial B_l}\right) + A_l\left(\frac{\partial H^{(\alpha-1)}_{\beta-1,l}}{\partial A_l} - \frac{\partial H^{(\alpha-2)}_{\beta-1,l}}{\partial B_l}\right) = 0.$$

The case $\alpha = \beta$ is again a matter of simple computation. □

Proposition 3.2. *The Hamiltonians*

$$K^{(\beta-k)}_{\beta,l} = \sum_{j=0}^{k-1}\binom{k-1}{j}(l-2)^{k-j-1}H^{(\beta-j-1)}_{\beta,l}$$
$$K^{(\beta)}_{\beta,l} = H^{(\beta)}_{\beta,l} \qquad l = 2,\ldots,N \tag{3.11}$$

satisfy the standard Lenard–Magri relations $P\,\mathrm{d}K^{(\alpha-1)}_{\beta,l} = R\,\mathrm{d}K^{(\alpha)}_{\beta,l}$, $\alpha = 1,\ldots\beta, \forall\beta$.

Proof. Using lemma 3, we have

$$
\begin{aligned}
R\,\mathrm{d}K_{\beta,l}^{(\beta-k)} &= P\,\mathrm{d}\left(\sum_{j=0}^{k-1}\binom{k-1}{j}\left((l-2)^{k-j-1}H_{\beta,l}^{(\beta-j-2)}+(l-2)^{k-j}H_{\beta,l}^{(\beta-j-1)}\right)\right)\\
&= P\,\mathrm{d}\left(K_{\beta,l}^{(\beta-k-1)}+\sum_{j=1}^{k-1}\left[\binom{k-1}{j-1}+\binom{k-1}{j}\right](l-2)^{k-j}H_{\beta,l}^{(\beta-j-1)}\right)\\
&= P\,\mathrm{d}\left(\sum_{j=0}^{k}\binom{k}{j}(l-2)^{k-j}H_{\beta,l}^{(\beta-j-1)}\right) = P\,\mathrm{d}K_{\beta,l}^{(\beta-k-1)}.
\end{aligned}
$$
□

3.3. Complete integrability for $\mathfrak{g}=sl(r)$

In this section we will prove that in the case $\mathfrak{g}=sl(r)$, the Hamiltonians (3.5) together with additional integrals one can recover from the global $SL(r)$ invariance of the model, define a completely integrable Hamiltonian system. We start by remarking that the content of lemma 3 and proposition 3.2 can be rephrased as follows: if we introduce the N matrices:

$$L_1 = A_1 \qquad L_a = \lambda A_a + B_a \qquad a = 2,\ldots,N \tag{3.12}$$

then they evolve isospectrally along any of the vector field of the hierarchy, that is (since the matrices A_i are generically simple) along Lax-type equations.

The dimension of the manifold $M = sl(r)^N$ is $d_M = (r^2-1)N$, and the dimension of the generic symplectic leaf of P is $d_S = d_M - N(r-1) = r(r-1)N$. We note that we recover (as expected) all the Casimirs of P considering: (a) the spectral invariants of $L_1 = A_1$ (this gives $N-1$ *common* Casimirs), and (b) the higher order terms in expansions of L_l: $H_{\alpha,l}^{(\alpha)}, \alpha = 2,\ldots,r, l = 2,\ldots,N$. Since it holds that $H_{\beta,l}^{(0)} = \sum_{k=0}^{\beta} H_{\beta,l-1}^{(k)}$, we consider the set

$$H_{\beta,l}^{(\alpha)} \quad \beta = 2,\ldots,r \qquad \alpha = 1,\ldots,\beta-1.$$

This provides us with a distinguished sequence of $\frac{r(r-1)}{2}$ mutually commuting Hamiltonians. Clearly, if $l \neq l'$, the sets $\{H_{\beta,l}^{(\alpha)}\}$ and $\{H_{\beta,l'}^{(\alpha)}\}$ are functionally independent, since they depend on a different set of variables. So, the counting of the number of independent Hamiltonians boils down to computing the counting of independent coefficients in the determinant

$$\det(\mu - \lambda A + B) \qquad A, B \in sl(r).$$

This problem was solved in [7], (see, also, [1]) and, actually, the number is exactly $\frac{r(r-1)}{2}$. Hence, the Lenard sequences associated with $R - \lambda P$ provide us with a total number of $(N-1)\frac{r(r-1)}{2}$ commuting Hamiltonians, plus the $N(r-1)$ Casimirs. For complete integrability we are missing $r(r-1)/2$ more commuting integrals.

They are associated with the global $SL(r)$ invariance of the problem, and, in the bi-Hamiltonian picture, can be described as follows. For every $\tau \in sl(r)$ we can consider the linear function

$$H_\tau = \mathrm{Tr}\left(\sum_{i=1}^{N} A_i \tau\right).$$

The Lenard 'sequence' associated with such functions is somewhat peculiar; indeed, since $R\,\mathrm{d}H_\tau = (N-1)P\,\mathrm{d}H_\tau$, we can associate with each H_τ a Lenard diagram which is (up to a constant) a closed loop, to be compared with the usual ladder typical of iterable Hamiltonians. However, the usual argument of bi-Hamiltonian recurrence, shows that, for any τ,

$$\left\{K_{\beta,l}^{(\alpha)}, H_\tau\right\}_P = \left\{K_{\beta,l}^{(\alpha)}, H_\tau\right\}_R = 0 \qquad \forall\, \alpha, \beta, l.$$

Indeed, one has, e.g., the equality:

$$\left\{K_{\beta,l}^{(\alpha)}, H_\tau\right\}_P = \left\{K_{\beta,l}^{(\alpha+1)}, H_\tau\right\}_R = (N-1)\cdot\left\{K_{\beta,l}^{(\alpha+1)}, H_\tau\right\}_P.$$

This argument shows how to recover, in the bi-Hamiltonian formalism, the integrals associated with the global $SL(r)$ invariance of the model. Clearly, this family of $r^2 - 1$ integrals is not a commutative one.

To recover the maximal Abelian subalgebra inside the Poisson algebra generated by the functions H_τ, one can consider (see, e.g., [6]):

(a) The $r-1$ independent elements $H_{h_1}, \ldots, H_{h_{r-1}}$ associated with, say, the standard Cartan subalgebra of $sl(r)$;

(b) The Gel'fand–Zetlin invariants, that is, the Casimirs of the nested subalgebras

$$sl(2) \subset sl(3) \subset \cdots \subset sl(r) \tag{3.13}$$

under the map $sl(r)^N \to sl(r)$ sending the N-tuple $\{A_1, \ldots, A_N\}$ into the total sum, $A_{\text{tot}} = \sum_{i=1}^N A_i$.

Noting that the Gel'fand–Zetlin functions corresponding to the last element of the chain (3.13) are given by $\sum_{k=0}^{\beta} H_{\beta,N}^{(k)}$, we obtain $r-1+\sum_{i=2}^{r-1}(i-1) = \frac{r(r-1)}{2}$ additional commuting integrals, which is exactly the number of commuting integrals we were looking for to ensure complete integrability of the model.

We end this section with a comment concerning super-integrability of the model. To this end we remark that we have at our disposal two Poisson pencils to construct families of commuting integrals for the Gaudin (homogeneous) Hamiltonian H_G: the pencil $R - \lambda P$ and the pencil[3] $Q - \lambda P$, described in section 2. On the $d_{N,r} = N(r(r-1))$-dimensional generic symplectic leaves of P they give rise to two distinct $d_{N,r}/2$ families of integrals of the motion K_m^{lj} and $\tilde{K}_m^{lj}$. Direct computations (which we performed for $r = 3, 4$ and $N \leqslant 6$) suggest that the number of functionally independent elements in the union of the two families is $d_{N,r} - (r-1)$. In other words, also taking into account the integrals coming from the global $SL(r)$ invariance of the model, we have super-integrability for the $sl(r)$ Gaudin model, that, however, is maximal only for the $sl(2)$ case.

4. Separation of variables for the *sl*(2) case

We consider now the N-particle $sl(2)$ case. The aim is to show that the Hamilton–Jacobi equations associated with the Hamiltonians

$$H_a = \operatorname{Tr}\left(A_a \cdot \sum_{j=1}^{a-1} A_j\right) \qquad a = 2, \ldots, N \tag{4.1}$$

and, in particular, the Hamilton–Jacobi equations associated with the physical Hamiltonian $H_G = \frac{1}{2}\sum_{i=1}^N H_i$ are separable in a very 'simple' set of coordinates. Our analysis is based on the so-called bi-Hamiltonian scheme for separation of variables (SoV), recently introduced in the literature (see, e.g., [22, 2, 8]). In particular, we will use the results for systems with an arbitrary number of (anchored) Lenard chains exposed in [10].

[3] With the proviso in mind that one has to set $\kappa = 0$ in those formulae.

We consider the manifold $M = sl(2)^N$, endowed with the Poisson pencil $R - \lambda P$, explicitly parametrized with the N matrices

$$A_i = \begin{bmatrix} h_i & f_i \\ e_i & -h_i \end{bmatrix}. \tag{4.2}$$

The generic symplectic leaf S of P is a $2N$-dimensional symplectic manifold, defined by the equations

$$C_i = \tfrac{1}{2}\operatorname{Tr} A_i^2 = h_i^2 + e_i f_i \qquad i = 1, \ldots, N$$

and can be (generically) endowed with the $2N$ coordinates $\{h_i, f_i\}_{i=1,\ldots,N}$.

According to the bi-Hamiltonian scheme [9], one modifies the tensor R in order to obtain a second compatible structure on S. Let us define the N vector fields

$$Z_i = \frac{1}{f_i}\frac{\partial}{\partial e_i} \tag{4.3}$$

and, for any pair of functions F, G on M, the following brackets:

$$\{F, G\}_{R'} = \{F, G\}_R - \sum_{a=2}^{N}(\{F, H_a\}_P Z_a(G) - \{G, H_a\}_P Z_a(F)) \tag{4.4}$$

where $X(F)$ denotes the Lie derivative of F with respect to the vector field X.

Thanks to the specific form of the vector fields Z_a, these new brackets restrict to S and give rise to *Poisson brackets* on S, which are compatible with the ones naturally induced by P. So, S is a symplectic manifold with respect to the restriction of the brackets associated with P, and is endowed with a (1, 1) tensor $\mathcal{N}$, with vanishing Nijenhuis torsion defined by

$$\mathcal{N} = R' \circ P^{-1}.$$

In such a geometrical setting, the bi-Hamiltonian scheme for SoV considers sets of coordinates $\{u_i, v_i\}$ (called Nijenhuis coordinates) associated with the eigenvalues λ_i of $\mathcal{N}$, characterized by the equations:

$$\mathcal{N}^* \mathrm{d}u_i = \lambda_i \,\mathrm{d}u_i \qquad \mathcal{N}^* \mathrm{d}v_i = \lambda_i \,\mathrm{d}v_i \tag{4.5}$$

whose Poisson brackets attain the remarkable form [21, 15]:

$$\begin{aligned} &\{u_i, u_j\}_P = \{u_i, u_j\}_{R'} = \{v_i, v_j\}_P = \{v_i, v_j\}_{R'} = 0 \\ &\{u_i, v_j\}_P = \delta_{ij}\vartheta_i(u_i, v_i) \qquad \{u_i, v_j\}_{R'} = \lambda_i\{u_i, v_j\}_P \end{aligned} \tag{4.6}$$

for some functions $\vartheta_i(u_i, v_i)$.

We shall prove the above statements directly displaying a set of Nijenhuis coordinates on the symplectic leaves of P.

Proposition 4.1. *The* $2N$ *functions*

$$\begin{aligned} &\lambda_1 = \sum_{i=1}^{N} f_i \qquad \lambda_a = -\frac{\sum_{k=1}^{a-1} f_k}{f_a} + (a-2) \qquad a = 2, \ldots, N \\ &\mu_1 = \sum_{i=1}^{N} h_i \qquad \mu_a = (\lambda_a - (a-2))h_a + \sum_{k=1}^{a-1} h_k \qquad a = 2, \ldots, N \end{aligned} \tag{4.7}$$

provide a set of Nijenhuis coordinates on S. In particular, the $\lambda_a, a = 2, \ldots, N$ *are the non-vanishing eigenvalues of* $\mathcal{N}^*$*, while* λ_1 *and* μ_1 *span its (two-dimensional) kernel.*

Proof. According to equation (4.4) and the definition of the Nijenhuis tensor $\mathcal{N}$, noting that both P and R' restrict to S and that $Z_a(\lambda_i) = Z_a(\mu_i) = 0$ for $a = 2, \ldots, N$, $i = 1, \ldots, N$, we have to show that, for any coordinate x_i, (that is, $x_i = e_i, h_i, f_i, i = 1, \ldots, N$) it holds that

$$\begin{aligned}
&\{x_i, \lambda_1\}_R - \sum_{a=2}^{N} \{H_a, \lambda_1\}_P Z_a(x_i) = 0 \\
&\{x_i, \lambda_b\}_R - \sum_{a=2}^{N} \{H_a, \lambda_b\}_P Z_a(x_i) - \lambda_b \{x_i, \lambda_b\}_P, b = 2, \ldots, N
\end{aligned} \tag{4.8}$$

as well as

$$\begin{aligned}
&\{x_i, \mu_1\}_R - \sum_{a=2}^{N} \{H_a, \mu_1\}_P Z_a(x_i) = 0 \\
&\{x_i, \mu_b\}_R - \sum_{a=2}^{N} \{H_a, \mu_b\}_P Z_a(x_i) = \lambda_b \{x_i, \mu_b\}_P, b = 2, \ldots, N.
\end{aligned} \tag{4.9}$$

The proof that these equations hold true is a matter of direct computations. One simply has to plug the explicit expressions of the Poisson brackets

$$\begin{aligned}
&\{h_i, e_j\}_P = \delta_{ij} e_j \qquad \{h_i, f_j\}_P = -\delta_{ij} f_j \qquad \{e_i, f_j\}_P = 2\delta_{ij} h_j \\
&\{h_i, e_j\}_R = \delta_{ij} \left[(i-1)e_i - \sum_{k=1}^{i-1} e_k \right] + \theta_{(i-j)} e_j + \theta_{(j-i)} e_i \\
&\{h_i, e_j\}_R = -\delta_{ij} \left[(i-1)f_i - \sum_{k=1}^{i-1} f_k \right] - \theta_{(i-j)} f_j - \theta_{(j-i)} f_i \\
&\{e_i, f_j\}_R = 2 \left\{ \delta_{ij} \left[(i-1)h_i - \sum_{k=1}^{i-1} h_k \right] + \theta_{(i-j)} h_j + \theta_{(j-i)} h_i \right\}
\end{aligned}$$

into equations (4.8) and (4.9), and use the identities

$$\sum_{j=1}^{n-1} \delta_{ij} = \theta_{(n-i)} \qquad \sum_{j=1}^{n-1} \theta_{(j-i)} F_j = \theta_{(n-i)} \sum_{j=i+1}^{n-1} F_j \tag{4.10}$$

$$\sum_{j=1}^{n-1} \theta_{(i-j)} F_j = (\theta_{(i-n)} + \delta_{in}) \sum_{j=1}^{n-1} F_j + \theta_{(n-i)} \sum_{j=1}^{i-1} F_j. \tag{4.11}$$

For example, let us consider $x_i \equiv h_i$. Since $Z_a(h_i) = 0$, we have (for $n \geqslant 2$):

$$\{h_i, \lambda_n\}_P = (\lambda_n - n + 2)\delta_{in} + \theta_{(n-i)} \frac{f_i}{f_n}$$

and

$$\begin{aligned}
\{h_i, \lambda_n\}_R &= \frac{1}{f_n} \sum_{j=1}^{n-1} \left\{ \delta_{ij} \left[(i-1)f_i - \sum_{k=1}^{i-1} f_k \right] + \theta_{(i-j)} f_j + \theta_{(j-i)} f_i \right\} \\
&\quad + \frac{\lambda_n - n + 2}{f_n} \left\{ \delta_{in} \left[(i-1)f_i - \sum_{k=1}^{i-1} f_k \right] + \theta_{(i-n)} f_n + \theta_{(n-i)} f_i \right\} \\
&= \lambda_n \left(\delta_{in}(\lambda_n - n + 2) + \theta_{(n-i)} \frac{f_i}{f_n} \right)
\end{aligned}$$

thanks to identities (4.10), (4.11).

The other cases of equations (4.8), (4.9) are proved with similar computations. □

To construct a set of *canonical* Nijenhuis coordinates (usually considered in the bi-Hamiltonian scheme for SoV and quite naturally termed *Darboux–Nijenhuis coordinates*) $\{\lambda_i, \phi_i\}_{i=1,\dots,N}$ from the Nijenhuis coordinates $\{\lambda_i, \mu_i\}_{i=1,\dots,N}$ one notes that an explicit computation gives

$$\{\lambda_1, \mu_1\} = -\lambda_1 \qquad \{\lambda_a, \mu_a\} = (\lambda_a - (a-2))(\lambda_a - (a-1)) \qquad a = 2, \dots, N. \tag{4.12}$$

Indeed, the first equation is trivially verified; for the remaining set of $N-1$ relations one has

$$\begin{aligned}\{\mu_n, \lambda_n\} &= -\frac{1}{f_n}\left\{\sum_{i=1}^{n-1} h_i, \sum_{i=1}^{n-1} f_i\right\} - (\lambda_n - (n-2))\{h_n, 1/f_n\}\sum_{i=1}^{n-1} f_i \\ &= \frac{\sum_{i=1}^{n-1} f_i}{f_n} - (\lambda_n - (n-2))\frac{\sum_{i=1}^{n-1} f_i}{f_n} = (\lambda_n - (n-2))(\lambda_n - (n-1)).\end{aligned}$$

Hence, one can choose

$$\phi_1 = -\frac{\sum_{i=1}^N h_i}{\sum_{i=1}^N f_i} \qquad \phi_a = \frac{\mu_a}{(\lambda_a - (a-2))(\lambda_a - (a-1))} \qquad a = 2, \dots, N \tag{4.13}$$

to have, together with the λ_i a set of Darboux–Nijenhuis coordinates.

To find the separation relations, we make contact with the so-called Sklyanin magic recipe [31]. To this end, we modify the Lax matrices (3.12), by a suitable shift in the spectral parameter λ. Namely we define

$$\tilde{L}_1 = L_1 \qquad \tilde{L}_a = (\lambda - (a-2))A_a + \sum_{b=1}^{a-1} A_b \qquad a = 2, \dots, N. \tag{4.14}$$

We note that the spectral invariants of $\tilde{L}_a$ are combinations of the Hamiltonians $H^{(\alpha)}_{\beta,l}$ and of the common Casimirs we considered in section 3.3, and provide an equivalent set of involutive constants of the motion (together with $H_1 = \sum h_i$).

As one can easily note, the Nijenhuis coordinates $\lambda_a, a = 2, \dots, N$ are nothing but the zeroes of the $(1, 2)$ entry of the Lax matrix $\tilde{L}_a$ of equation (4.14), while the Nijenhuis coordinates $\mu_a, a = 2, \dots, N$ are the values for $\lambda = \lambda_a$ of the $(1, 1)$ entry of the same matrix. Taking into account that μ_1 is the first Hamiltonian H_1, we see that the Nijenhuis coordinates, the Hamiltonians and the Casimirs $C_i = \mathrm{Tr}(A_i)^2$ are related by the separated equations:

$$\mu_1 - H_1 = 0 \qquad \mathrm{Det}\left(\mu_a - \left((\lambda_a - (a-2))A_a + \sum_{i=1}^{a-1} A_i\right)\right) = 0 \tag{4.15}$$

whence one can directly find, using relations (4.13), the 'canonical' separation relations for the Hamilton–Jacobi equation associated with the $sl(2)$ Gaudin Hamiltonians. We note that these are quadratic equations in the separated coordinates, and hence explicitly solvable by elementary functions, for every number of sites N.

5. Summary and comments

In this paper, we have used tools from bi-Hamiltonian geometry to study the integrability of the classical rational XXX Gaudin models, associated with the Lie algebra $sl(r)$. We first framed the general (that is, inhomogeneous) model within the Gel'fand–Zakharevich scheme by selecting a suitable pencil of Poisson brackets induced by a natural family of Poisson brackets on the space of matrix polynomials.

Then we extensively studied the *homogeneous case*. We considered an alternative complete set of mutually commuting constants of the motion I_k (independent of the parameters usually entering the formulation of Gaudin models). These integrals (actually, a subfamily thereof), in the $su(2)$ case, coincide with the Hamiltonians of the bending flows of Kapovich and Millson on the moduli space of polygons in the Euclidean space.

We introduced an 'additional' Poisson tensor which forms, together with the standard Lie–Poisson tensor, a bi-Hamiltonian pencil. The GZ analysis of such a bi-Hamiltonian structure provides exactly the alternative set of constants of the motion I_k for the $sl(2)$ case. By using such a bi-Hamiltonian scheme, we extended this analysis to the $sl(r)$ case; in particular, we show that the higher rank counterparts of the additional set of integrals guarantee complete integrability of the $sl(r)$-Gaudin magnet. Since, we still have at our disposal the Jurčo–Sklyanin integrals, we conclude that the Gaudin magnet is super-integrable (although we could not establish *maximal super-integrability*) also in the $sl(r)$ case.

We furthermore have explicitly shown in the $sl(2)$ case that the Hamilton–Jacobi equations associated with the set of additional integrals can be solved by separation of variables, using the bi-Hamiltonian scheme for SoV which has recently been considered in the literature. Actually, what we found is a set of separation coordinates *alternative* to the 'standard' one found by Sklyanin and the 'Montreal group', based on the standard Lax representation for the (homogeneous) Gaudin model. This should not be regarded as a surprise, since it is well known that super-integrability is related to the existence of different sets of separation coordinates. In this set of coordinates, the Hamilton–Jacobi equations can be explicitly solved by elementary functions, or, otherwise stated, the separation coordinates (for the $sl(2)$-magnet) live on genus 0 spectral curves for *any* number of particles, while, in the 'standard picture', the genus of the spectral curve grows linearly with N.

This result, and in particular the fact that separation coordinates are *rational* functions of the 'physical' coordinates can be seen as the counterpart of the fact that the model is, although by means of a non-completely trivial transformation, amenable to the study of nested subsystems, each of those is equivalent to a two-particle system [4]. This is particularly clear in the $su(2)$ quantum case, where the spectra of the commuting parameter-independent Hamiltonians can be computed by means of Lie-algebraic methods (namely, the representation theory of $su(2)$) [23, 24].

Finally we would like to add the following comment. We are not in a position yet to make a clearcut connection between the bi-Hamiltonian structure $R - \lambda P$ we introduced in the present paper, and the solution of the quantum (homogeneous) $sl(r)$ Gaudin models. However, the fact that, for the classical model, we managed to construct the complete family of involutive integrals (3.11) that admit a simple Lie-algebraic characterization, suggests that the involutivity property of such distinguished functions might survive quantization and provide a set of quantum integrals of motion whose diagonalization could possibly be obtained without resorting to Bethe ansatz techniques. Work in this direction is in progress.

Acknowledgments

We wish to thank J Harnad for many useful discussions, and, especially, for drawing our attention to the papers [18, 12] concerning the bending flows. Also, we thank B Dubrovin, F Magri, M Pedroni, and O Ragnisco for their interest in this work. Thanks are also due to the anonymous referees whose comments have helped to improve the presentation of the paper.

Appendix A

In this appendix we sketch the proof of lemma 1. Actually we will prove the converse statement, i.e., that the image of the Poisson tensor (2.11) under the map (2.8) is the diagonal Poisson tensor (2.6).

We denote with J the Jacobian of the transformation:

$$J_{ij} = \frac{\partial B_{i-1}}{\partial A_j} = (-1)^{N-i} s_{N-i}(a_1, \ldots, \hat{a}_j, \ldots, a_N). \tag{A.1}$$

Using the identity:

$$\sum_{j=1}^{N} x^{j-1}(-1)^{N-j} s_{N-j}(a_1, \ldots, \hat{a}_k, \ldots, a_N) = \prod_{l\neq k}(x - a_l) \tag{A.2}$$

the inverse matrix of (A.1) is easily obtained:

$$(J^{-1})_{ij} = \frac{a_i^{j-1}}{\prod_{k\neq i}(a_i - a_k)}. \tag{A.3}$$

We have

$$((J^{-1} P J^{-1})^t)_{in} = \frac{(-1)^N}{\prod_{m\neq i}(a_i - a_m)\prod_{p\neq n}(a_n - a_p)}\left(P_{in}^{(1)} - P_{in}^{(2)}\right) \tag{A.4}$$

with

$$P_{in}^{(1)} = \sum_{r=0}^{N-1}\sum_{l=r+1}^{N}(-1)^l s_{N-l}(a_1, \ldots, a_N)\sum_{k=r+1}^{l} a_i^{r+l-k} a_n^{k-1}[B_r, \cdot] \tag{A.5}$$

$$P_{in}^{(2)} = \sum_{r=1}^{N}\sum_{l=0}^{r-1}(-1)^l s_{N-l}(a_1, \ldots, a_N)\sum_{k=l+1}^{r} a_i^{r+l-k} a_n^{k-1}[B_r, \cdot]. \tag{A.6}$$

Subtracting (A.5) and (A.6), by using induction and the identities

$$\begin{aligned} &s_i(a_1, \ldots, a_{N+1}) = s_i(a_1, \ldots, a_N) + a_{N+1}s_{i-1}(a_1, \ldots, a_N) \\ &s_i(a_1, \ldots, a_N) = 0 \quad \text{if} \quad i < 0 \quad \text{or} \quad i > N \end{aligned} \tag{A.7}$$

$$\sum_{l=0}^{N}(-1)^l s_{N-l}(a_1, \ldots, a_N)x^l = (-1)^N\prod_{i=1}^{N}(x - a_i) \tag{A.8}$$

one proves that the coefficient of B_r in formula (A.4) vanishes if $i \neq n$.

Now let us consider the diagonal terms

$$P_{ii}^{(1)} - P_{ii}^{(2)} = \sum_{r=0}^{N}\left(a_i^r\sum_{l=0}^{N}(-1)^l s_{N-l}(a_1, \ldots, a_N)(l - r)a_i^{l-1}[B_r, \cdot]\right).$$

Since

$$\sum_{l=0}^{N}(-1)^l s_{N-l}(a_1, \ldots, a_N) l a_i^{l-1} = (-1)^N\frac{\mathrm{d}}{\mathrm{d}x}\left(\prod_j (x - a_j)\right)\Bigg|_{x=a_i} = (-1)^N\prod_{j\neq i}(a_i - a_j)$$

we get (using (2.8) and (A.2)):

$$P_{ii}^{(1)} - P_{ii}^{(2)} = (-1)^N \prod_{j\neq i}(a_i - a_j) \sum_{r=0}^{N} a_i^r [B_r, \cdot] = (-1)^N \left(\prod_{j\neq i}(a_i - a_j)\right)^2 [A_i, \cdot]$$

whence the assertion.

Appendix B

Lemma 4. *The Poisson tensors Q (2.12) and R (3.2) are not compatible for $N \geqslant 3$ for any choice of the parameters $a_1, \ldots, a_N$.*

Proof. An explicit computation shows that the Poisson tensor Q can be written in the 'coordinates' $\{A_1, \ldots, A_N\}$ as

$$\{F, G\}_Q = \sum_{i,j,k} q_{ijk} \operatorname{Tr}\left(A_k \left[\frac{\partial G}{\partial A_j}, \frac{\partial F}{\partial A_i}\right]\right)$$

with

$$q_{ijk} = (-1)^N \left\{\delta_{ij}\left(\xi_j \delta_{jk} + \beta_j \frac{(1-\delta_{jk})}{\eta_{jk}}\right) + \frac{(\beta_i \delta_{jk} - \beta_j \delta_{ik})}{\eta_{ji}}\right\}$$

$$\eta_{ij} = \begin{cases} a_i - a_j & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases} \qquad \beta_i = \prod_{k\neq i} \frac{a_k}{\eta_{ik}} \qquad \xi_i = \sum_{j\neq i} \frac{\beta_j}{\eta_{ji}}.$$

Using this expression it is easy to evaluate the Schouten bracket of Q, R on the differentials of the functions

$$F = \operatorname{Tr}(A_1 h) \qquad G = \operatorname{Tr}(A_2 x) \qquad H = \operatorname{Tr}(A_2 h)$$

where with x and h we denoted two constant matrices satisfying $[h, x] = x$. We have

$$[Q, R]_S(\mathrm{d}F, \mathrm{d}G, \mathrm{d}H) = (-1)^N \left[\xi_2 - \xi_1 + \frac{(\beta_1 - \beta_2)}{\eta_{21}} + \beta_2 \sum_{j=3}^{N} \frac{1}{\eta_{2j}}\right] \operatorname{Tr}(A_1 x) + \beta_1 \sum_{k=3}^{N} \frac{1}{\eta_{k1}} \operatorname{Tr}(A_k x). \tag{B.1}$$

A necessary condition for (B.1) to vanish is that $\beta_1 = 0$, i.e., one of the constants $a_2, \ldots, a_N$ must be equal to zero. Let us suppose $a_k = 0$, $k > 2$; then

$$[Q, R]_S(\mathrm{d}F, \mathrm{d}G, \mathrm{d}H) = (-1)^N \left(\frac{1}{a_2} - \frac{1}{a_1}\right) \operatorname{Tr}(A_1 x) \neq 0$$

since $a_1 \neq a_2$.

In the case $a_2 = 0$, we have instead

$$[Q, R]_S(\mathrm{d}F, \mathrm{d}G, \mathrm{d}H) = (-1)^N \operatorname{Tr}(A_1 x) \sum_{j=3}^{N} \frac{1}{a_j}. \tag{B.2}$$

If $N = 3$ then (B.2) is different from zero, so we can assume $N > 3$. But if $a_2 = 0$ and $N > 3$ we can consider

$$F' = \operatorname{Tr}(A_1 h) \qquad G' = \operatorname{Tr}(A_2 x) \qquad H' = \operatorname{Tr}(A_3 h).$$

Since

$$[Q, R]_S(\mathrm{d}F', \mathrm{d}G', \mathrm{d}H') = (-1)^{N+1}\frac{1}{a_3}\,\mathrm{Tr}(A_1 x) \neq 0$$

the proof is concluded. □

References

[1] Adams M, Harnad J and Hurtubise J 1993 Darboux coordinates and Liouville–Arnold integration in loop algebras *Commun. Math. Phys.* **155** 385–413

Adams M, Harnad J and Hurtubise J 1996 Darboux coordinates on coadjoint orbits of Lie algebras *Lett. Math. Phys.* **40** 41–57

[2] Błaszak M 1998 On separability of bi-Hamiltonian chain with degenerated Poisson structures *J. Math. Phys.* **39** 3213–35

[3] Ballesteros A and Ragnisco O 2002 Classical Hamiltonian systems with $sl(2)$ coalgebra symmetry and their integrable deformations *J. Math. Phys.* **43** 954–69

[4] Ballesteros A and Ragnisco O 2003 Classical dynamical systems from q-algebras:'cluster' variables and explicit solutions *Preprint* math-ph/0307013

[5] Ballesteros A, Corsetti M and Ragnisco O 1996 N-dimensional classical integrable systems from Hopf algebras *Czech. J. Phys.* **46** 1153–65

[6] Drozd Yu A, Ovsienko S A and Futorny V M 1991 On Gel' fand-Zetlin modules *Proc. Winter School on Geometry and Physics (Srni, 1990) Rend. Circ. Mat. Palermo (2) Suppl.* **26** 143–7

[7] Dubrovin B A 1985 Matrix finite-zone operators *J. Sov. Math.* **28** 20–50

[8] Falqui G, Magri F and Pedroni M 2001 Bihamiltonian geometry and separation of variables for Toda lattices *J. Nonlinear Math. Phys.* **8** 118–27

[9] Falqui G and Pedroni M 2002 On a Poisson reduction for Gel'fand–Zakharevich manifolds *Rep. Math. Phys.* **50** 395–407

[10] Falqui G and Pedroni M 2003 Separation of variables for bi-Hamiltonian systems *Math. Phys. Anal. Geom.* **6** 139–79

[11] Falqui G and Musso F 2002 Bi-Hamiltonian geometry and separation of variables for Gaudin models: a case study *Proc. Int. Conf. SPT 2002 Symmetry and Perturbation Theory* (Singapore: World Scientific) pp 42–50

[12] Flaschka H and Millson J 2001 The moduli space of weighted configurations on projective space *Preprint* math.SG/0108191

[13] Gaudin M 1976 Diagonalisation d' une classe d' hamiltoniens de spin *J. Physique* **37** 1087–98

See, also Gaudin M 1983 *La Fonction d' Onde de Bethe* (Paris: Masson)

[14] Gekhtman M I 1995 Separation of variables in the classical *SL(N)* magnetic chain *Commun. Math. Phys.* **167** 593–605

[15] Gel'fand I M and Zakharevich I 1993 On the local geometry of a bi-Hamiltonian structure. *The Gel'fand Mathematical Seminars 1990-1992* ed L Corwin *et al* (Boston: Birkhäuser) pp 51–112

Gel'fand I M and Zakharevich I 2000 Webs, Lenard schemes, and the local geometry of bihamiltonian Toda and Lax structures *Selecta Math. (N.S.)* **6** 131–83

[16] Harnad J and Yermolaeva O 2003 Super-integrability, Lax matrices and separation of variables *Preprint* nlin.SI/0303009

[17] Jurčo B 1989 Classical Yang–Baxter equations and quantum integrable systems *J. Math. Phys.* **30** 1289–93

[18] Kapovich M and Millson J 1996 The symplectic geometry of polygons in Euclidean space *J. Diff. Geom.* **44** 479–513

[19] Karimipour V 1998 Integrable structure of the new Calogero models *J. Math. Phys.* **39** 913–20

[20] Magri F and Morosi C 1986 Sulla relazione tra varietà bihamiltoniane ed i problemi spettrali della teoria dello scattering inverso *Atti* 8° *Congresso AIMETA (Torino, 1896)* vol II pp 675–9 (in Italian)

[21] Magri F 1990 Geometry and soliton equations *Atti. Acc. Sci. Torino Cl. Sci. Fis. Mat. Natur.* **124** 181–209

[22] Morosi C and Tondo G 1997 Quasi–bi–Hamiltonian systems and separability *J. Phys. A: Math. Gen.* **30** 2799–806

[23] Musso F and Ragnisco O 2000 Exact solution of the quantum Calogero–Gaudin system and of its q deformation *J. Math. Phys.* **41** 7386–401

[24] Musso F and Ragnisco O 2001 The spin-1/2 Calogero-Gaudin system and its q-deformation *J. Phys. A: Math. Gen.* **34** 2625–35

[25] Panasyuk A 2003 Projections of Jordan bi-Poisson structures that are Kronecker, diagonal actions, and the classical Gaudin systems *J. Geom. Phys.* **47** 379–97

[26] Pedroni M and Vanhaecke P 1998 A Lie algebraic generalization of the Mumford system, its symmetries and its multi-Hamiltonian structure *Regul. Chaotic Dyn.* **3** 132–60

[27] Reyman A G and Semenov-Tian-Shansky M A 1994 Group-theoretical methods in the theory of finite-dimensional integrable systems in dynamical systems terms VII (Berlin: Springer)

[28] Scott D R D 1994 Classical functional Bethe ansatz for $SL(N)$: separation of variables for the magnetic chain *J. Math. Phys.* **35** 5831–43

[29] Sklyanin E K 1992 Separation of variables in the classical integrable SL(3) magnetic chain *Commun. Math. Phys.* **150** 181–92

[30] Sklyanin E K 1989 Separation of variables in the Gaudin model *J. Sov. Math.* **47** 2473–88

[31] Sklyanin E K 1995 Separation of variables: new trends *Prog. Theor. Phys. Suppl.* **118** 35–60